

# Condition for the higher rank numerical range to be non-empty

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## Abstract

*It is shown that the rank- $k$  numerical range of every  $n$ -by- $n$  complex matrix is non-empty if  $k < n/3 + 1$ . The proof is based on a recent characterization of the rank- $k$  numerical range by Li and Sze, the Helly's theorem on compact convex sets, and some eigenvalue inequalities. In particular, the result implies that  $\Lambda_2(A)$  is non-empty if  $n \geq 4$ . This confirms a conjecture of Choi et al. If  $k \geq n/3 + 1$ , an  $n$ -by- $n$  complex matrix is given for which the rank- $k$  numerical range is empty. Extension of the result to bounded linear operators acting on an infinite dimensional Hilbert space is also discussed.*

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## 1 Introduction

Let  $M_n$  be the algebra of  $n \times n$  complex matrices. In [3], the authors introduced the notion of the rank- $k$  numerical range of  $A \in M_n$  defined and denoted by

$$\Lambda_k(A) = \{\lambda \in \mathbf{C} : X^*AX = \lambda I_k, \text{ } X \text{ is } n \times k \text{ such that } X^*X = I_k\}$$

in connection to the study of quantum error correction; see [4]. Evidently,  $\lambda \in \Lambda_k(A)$  if and only if there is a unitary matrix  $U \in M_n$  such that  $U^*AU$  has  $\lambda I_k$  as the leading principal submatrix. When  $k = 1$ , this concept reduces to the classical numerical range, which is well known to be convex by the Toeplitz-Hausdorff theorem; for example, see [7] for a simple proof. In [1] the authors conjectured that  $\Lambda_k(A)$  is convex, and reduced the convexity problem to the problem of showing that  $0 \in \Lambda_k(A)$  for

$$A = \begin{pmatrix} I_k & X \\ Y & -I_k \end{pmatrix}$$

for arbitrary  $X, Y \in M_k$ . They further reduced this problem to the existence of a Hermitian matrix  $H$  satisfying the matrix equation

$$I_k + MH + HM^* - HPH = H \quad (1)$$

for arbitrary  $M \in M_k$  and positive definite  $P \in M_k$ . In [10], the author observed that equation (1) can be rewritten as the continuous Riccati equation

$$HPH - H(M^* - I_k/2) - (M - I_k/2)H - I_k = 0_k, \quad (2)$$

and existing results on Riccati equation will ensure its solvability; for example, see [5, Theorem 4]. This establishes the convexity of  $\Lambda_k(A)$ .

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For a Hermitian  $X \in M_n$ , let  $\lambda_1(X) \geq \dots \geq \lambda_n(X)$  be the eigenvalues of  $X$ . In [8], it was shown that

$$\Lambda_k(A) = \{\mu \in \mathbf{C} : e^{it}\mu + e^{-it}\bar{\mu} \leq \lambda_k(e^{it}A + e^{-it}A^*) \text{ for all } t \in [0, 2\pi)\}. \quad (3)$$

In particular,  $\Lambda_k(A)$  is the intersection of closed half planes on  $\mathbf{C}$ , and therefore is always convex. Moreover, if  $A \in M_n$  is normal with eigenvalues  $\lambda_1, \dots, \lambda_n$ , then

$$\Lambda_k(A) = \bigcap_{1 \leq j_1 < \dots < j_{n-k+1} \leq n} \text{conv} \{\lambda_{j_1}, \dots, \lambda_{j_{n-k+1}}\}.$$

This confirms a conjecture in [2].

While many interesting results have been obtained for  $\Lambda_k(A)$ , see [1, 2, 3, 4] for example, there are some basic questions whose answers are unknown. The purpose of this paper is to answer the following.

**Problem** Determine  $n$  and  $k$  such that  $\Lambda_k(A)$  is non-empty for every  $A \in M_n$ .

It is well-known that the classical numerical range  $\Lambda_1(A)$  is non-empty. For  $k > n/2$ ,  $\Lambda_k(A)$  has at most one element and one can easily construct  $A \in M_n$  such that  $\Lambda_k(A) = \emptyset$ ; see Proposition 2.2 and Corollary 2.3 in [3]. The situation for  $\Lambda_k(A)$  with  $n/2 \geq k > 1$  is not so clear. In [2], the authors conjectured that  $\Lambda_2(A) \neq \emptyset$  for  $n \geq 4$ .

In the next section, we show that  $\Lambda_k(A)$  is non-empty for every  $A \in M_n$  if and only if  $k < n/3 + 1$ . In particular, it confirms the conjecture in [2] that  $\Lambda_2(A) \neq \emptyset$  if  $n \geq 4$ . We also consider extension of the result to infinite dimensional bounded linear operators.

## 2 Results and proofs

**Theorem 1** *Let  $A \in M_n$ , and let  $k$  be a positive integer such that  $k < n/3 + 1$ . Then  $\Lambda_k(A)$  is non-empty.*

*Proof.* Evidently,  $\Lambda_k(A) \subseteq \Lambda_1(A)$ . Given  $A \in M_n$  and  $t \in [0, 2\pi)$ , let  $A(t) = e^{it}A + e^{-it}A^*$ . Consider the compact convex sets

$$S(t) = \{\mu \in \Lambda_1(A) : e^{it}\mu + e^{-it}\bar{\mu} \leq \lambda_k(A(t))\}, \quad t \in [0, 2\pi).$$

By (3),

$$\Lambda_k(A) = \bigcap_{t \in [0, 2\pi)} S(t).$$

By Helly's Theorem [6, Theorem 24.9], it suffices to show that  $S(t_1) \cap S(t_2) \cap S(t_3) \neq \emptyset$  for all choices of  $t_1, t_2, t_3$  with  $0 \leq t_1 < t_2 < t_3 < 2\pi$ .

For  $1 \leq j \leq 3$ , let  $V_j$  be the subspace spanned by the eigenvectors of  $A(t_j)$  corresponding to the eigenvalues  $\lambda_k(A(t_j)), \dots, \lambda_n(A(t_j))$ . Then  $\dim V_j \geq n - k + 1$ . Hence, we have

$$\begin{aligned} & \dim(V_1 \cap V_2 \cap V_3) \\ &= \dim(V_1 \cap V_2) + \dim V_3 - \dim((V_1 \cap V_2) + V_3) \\ &= \dim V_1 + \dim V_2 - \dim(V_1 + V_2) + \dim V_3 - \dim((V_1 \cap V_2) + V_3) \\ &\geq 3(n - k + 1) - 2n \\ &= n - 3k + 3 \\ &\geq 1. \end{aligned}$$

Let  $v$  be a unit (column) vector in  $V_1 \cap V_2 \cap V_3$ . Then  $\mu = v^*Av \in \Lambda_1(A)$  and for  $1 \leq j \leq 3$ , we have

$$e^{it}\mu + e^{-it}\bar{\mu} = v^*(A(t_j))v \leq \lambda_k(A(t_j)).$$

Hence,  $\mu \in S(t_1) \cap S(t_2) \cap S(t_3)$ . □

The following answers a question in [2].

**Corollary 2** *Let  $A \in M_n$  with  $n \geq 4$ . Then  $\Lambda_2(A) \neq \emptyset$ .*

Without additional information on  $A \in M_n$ , the bound on  $n$  in Theorem 1 is best possible as shown by the following result.

**Theorem 3** *Suppose  $k$  is a positive integer such that  $k \geq n/3 + 1$ . There exists  $A \in M_n$  such that  $\Lambda_k(A) = \emptyset$ .*

*Proof.* We first consider the case when  $3k = n + 3$ . Let  $w = e^{i2\pi/3}$ , and

$$A = I_{k-1} \oplus wI_{k-1} \oplus w^2I_{k-1}.$$

Write  $A = H + iG$  with  $H = H^*$  and  $G = G^*$ . Then  $H = I_{k-1} \oplus (-1/2)I_{2k-2}$ . Thus,  $\Lambda_k(H) = \{-1/2\}$ ; see also [3, Theorem 2.4]. So,

$$\Lambda_k(A) \subseteq \mathcal{L} = \{z : \operatorname{Re} z = -1/2\}.$$

By rotation of  $2\pi/3$  and  $4\pi/3$ , one can show that  $\Lambda_k(A) \subseteq w\mathcal{L}$  and  $\Lambda_k(A) \subseteq w^2\mathcal{L}$ . So,

$$\Lambda_k(A) \subseteq \mathcal{L} \cap w\mathcal{L} \cap w^2\mathcal{L} = \emptyset.$$

Now, suppose  $3k > n + 3$ . Then we can consider a principal submatrix  $B \in M_n$  of the matrix  $A \in M_{3k-3}$  constructed in the preceding paragraph. Then  $\Lambda_k(B) \subseteq \Lambda_k(A) = \emptyset$ . □

Note that we can perturb the example in the above proof to get a non-normal matrix  $A \in M_n$  such that  $\Lambda_k(A) = \emptyset$  if  $k \geq n/3 + 1$ . Also, Theorem 3 can be obtained from parts (1), (2), (3) of [2, Theorem 4.7] and the fact that  $\Lambda_k(A)$  is a subset of

$$\bigcap_{1 \leq j_1 < \dots < j_{n-k+1} \leq n} \operatorname{conv} \{\lambda_{j_1}, \dots, \lambda_{j_{n-k+1}}\}$$

if  $A \in M_n$  is normal with eigenvalues  $\lambda_1, \dots, \lambda_n$ .

Let  $\mathcal{B}(\mathcal{H})$  be the algebra of bounded linear operator acting on an infinite dimensional Hilbert space  $\mathcal{H}$ . One can extend the definition of  $\Lambda_k(A)$  for a bounded linear operator  $A \in \mathcal{B}(\mathcal{H})$  by

$$\Lambda_k(A) = \{\gamma \in \mathbf{C} : X^*AX = \gamma I_k, X : \mathbf{C}^k \rightarrow \mathcal{H}, X^*X = I_k\}.$$

By Theorem 1, we have the following.

**Corollary 4** *Suppose  $k$  is a positive integer and  $A \in \mathcal{B}(\mathcal{H})$  for an infinite dimensional Hilbert space  $\mathcal{H}$ . Then*

$$\Lambda_k(A) \neq \emptyset.$$

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